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High Frequency Sound According To the Boltzmann Equation

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Abstract

The boundary value problem of sound generated at an oscillating wall propagating into a half space is solved for a linear Boltzmann equation with a general cutoff intermolecular potential and an arbitrary boundary condition at the wall. For a small bounded domain (two walls closer than a mean free path) proof of the existence of a solution is relatively simple. In a half space the solution is proved to exist for sufficiently high frequency $\omega > \omega_o$ (ω_o is comparable to the mean collision frequency). Asymptotic expressions show the disappearance of a conventional sound wave at high frequency. This is consistent with known results for a relaxation model of the Boltzmann equation and qualitative estimates for the actual Boltzmann equation based on the dominance of the continuous spectrum at high frequency. At a fixed arbitrary distance from the wall, the dominant high frequency disturbance consists of just those particles emitted by the wall, decreased by scattering, but without any compensating contribution from the creation term of the collision operator.

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1. Introduction

The propagation of sound waves from an oscillating wall into a gas has been treated by many approximate methods, but no satisfactory mathematical solution has been given for the Boltzmann equation. An elegant solution to this problem for a model of the Boltzmann equation has been obtained by Weitzner [1]. There are several high order moment and polynomial approximations [2], [3], [4]; but it is difficult to assess their accuracy since they are essentially *ad hoc* and insensitive to the many singular features of the solution. An elaborate model of the Boltzmann equation has been used by Sirovich and Thurber [5] but only to evaluate normal modes, not to solve the boundary value problem. A polynomial approximation in the neighborhood of free flow has been described which is much more likely to give an accurate representation in the high

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1. Harold Weitzner, "Steady-State Oscillations in a Gas," to appear in Proceedings of the Fourth International Symposium on Rarefied Gas Dynamics, Toronto, 1964.

2. C. S. Wang Chang and G. Uhlenbeck, "Propagation of Sound in Monatomic Gases," Engineering Research Institute, University of Michigan, 1952.

3. L. Sirovich, Physics of Fluids 6, 218 (1963).

4. C. L. Pekeris, Z. Alterman, L. Finkelstein, and K. Frankowski, Physics of Fluids 5, 1608 (1962).

5. L. Sirovich, J. K. Thurber, Journal of the Acoustical Society 37, 329 (1965).

frequency regime [6]; but again the error is hard to estimate and not all singular features are visible.

There are two major questions: the existence of a solution to the problem, a decidedly nontrivial matter; and descriptive properties of the solutions which are emphatically singular, especially at high frequency.

With regard to existence, there has been a substantial development for the initial value problem (with trivial boundaries) [7], even to some extent in nonlinear problems [8]. The basic tool in this theory is the recognition that a certain class of cutoff intermolecular potentials yields a mathematically accessible Boltzmann equation. But certain *a priori* estimates which are crucial for solution in the large in the initial value theory are not available in boundary value problems. For this reason only existence in the small, e.g., between two walls which are sufficiently close, is an easy matter. We use the high frequency, $\omega > \omega_o$, to improve our estimates enough to extend the solution to an entire half space.

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6. D. Kahn and D. Mintzner, Physics of Fluids 8, 1090 (1965).

7. Harold Grad, "Asymptotic Theory of the Boltzmann Equation, II" in Rarefied Gas Dynamics, ed. J. A. Laurmann, Academic Press (1963).

8. Harold Grad, "Asymptotic Equivalence of the Navier-Stokes and Nonlinear Boltzmann Equations," in Proceedings of Symposia in Applied Mathematics, Vol. 17, American Mathematical Society (1965).

But the question of solution of even the steady flow linear Boltzmann equation in sizable domains is largely untouched. The problem of making plausible rather than rigorous asymptotic estimates (e.g., near the wall, far away, at very high frequency, for low or high speed particles) is not so difficult and is not significantly harder for the actual Boltzmann equation than it is for a model equation.

The main qualitative feature of all problems in kinetic theory is the transition from free flow to continuum flow. For the initial value problem an essentially complete answer is known [9], [10]. For a very short time there is an approximate free flow; this is followed by a readjustment to an approximately local Maxwellian on a time scale comparable to the mean collision time; then there is a much longer era of macroscopic fluid motion and an ultimate decay to equilibrium. Similarly, in a steady boundary value problem we can expect something like free flow (characterized by two distinct half-distributions) very close to the wall; a kinetic boundary layer on the order of the mean free path; and a subsequent fluid regime governed by macroscopic equations. But instead of a distant behavior which is fluid-like, there is (at least for moderate frequencies) only an intermediate zone in which

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9. Harold Grad, Physics of Fluids 6, 147 (1963).
10. Harold Grad, Journal SIAM 13, 259 (1965).

a sound wave can be perceived. This is followed by a complex kinetic ultimate decay (discovered for the model equation by Weitzner [1]). This ultimate kinetic regime also exists in the initial value problem [7] for a class of soft potentials which is unrealistic in neutral gases; in the boundary value problem this effect is more common.

The reason for this phenomenon is the presence of a continuous spectrum extending to the origin. This fact alone provides a simple qualitative description even for the actual Boltzmann equation [11]. The non-exponentially decaying contribution from the continuum ultimately dominates, even though the more highly excited normal mode is initially more important. But for sufficiently high frequency the normal mode (i.e., sound wave) is completely submerged at any distance from the wall.

In a boundary value problem in which there is algebraic rather than exponential decay at infinity, such as flow around a body, the effect of the continuum may not be evident [12].

Another source of unusual and singular behavior arises

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11. Harold Grad, "Theory of the Boltzmann Equation," to appear in Proceedings of the Eleventh International Congress of Applied Mechanics, Munich, 1964.

12. Harold Grad, "Flow Equations in a Rarefied Atmosphere," in Proceedings of Symposium on Aerodynamics of the Upper Atmosphere, Project Rand (1959).

from the very slow and very fast particles. The former are disruptive because the free path is arbitrarily small. There can be very rapid spatial gradients, and at the wall itself infinite gradients (see Sec. 3). This is perhaps the most serious hindrance to the development of a general existence theory. The fast particles are characterized by unbounded free paths (except in the special case of hard spheres), and they dominate the behavior far from the wall; this gives a more intuitive picture of the origin of the continuous spectrum. An important quantitative point, that the collision frequency is velocity dependent, makes it quite important to consider the actual Boltzmann equation rather than a model.

2. The Spectrum

The distinction between a discrete and a continuous spectrum is basically mathematical, but the qualitative difference in the behavior of solutions is so great as to yield very important qualitative physical differences. In a normal mode all the essential properties of a wave belong to the gas alone (e.g., the phase speed and decrement); the initial and boundary values only serve

to determine the level of excitation. But a mode which results from a continuum never loses sight of the entire initial or boundary data; it does not have a definite speed or a definite wavelength, nor does it decay exponentially. And each of these properties is inextricably interwoven in characteristics of both the medium and the boundary. Thus, although one can give mathematical formulations which combine both possibilities within a common framework, it is necessary to distinguish the two in order to understand the physical consequences.

There are two sources of continuous spectra in kinetic theory. The first and more obvious results from streaming. The operator $\xi \cdot \partial f/\partial x$ has the entire imaginary axis as its spectrum. In the form

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} = 0 \tag{2.1}$$

(free flow) we recognize that there are no normal modes. Yet all macroscopic moments decay to equilibrium as $t \longrightarrow \infty$, not at a predetermined rate, but at a rate which depends on the initial data [13].

The second source of a continuum is in the collision operator L. This property has only recently been recognized and is found to accompany almost all intermolecular

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13. Harold Grad, Communications on Pure and Applied Mathematics 14, 323 (1961).

potentials [7]. In the form

$$\frac{\partial f}{\partial t} = L[f] \tag{2.2}$$

(spatially uniform) we observe no simple exponential decay to equilibrium; but for the most interesting case of hard potentials, all solutions do decay faster than a given exponential.

The spectrum in a more complicated problem reflects the interaction between the streaming operator and the collision operator. We write the full linear Boltzmann equation in the form

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} = L[f] \tag{2.3}$$

where

$$L[f] = K[f] - \nu f \ . \tag{2.4}$$

L is the linearized collision operator and is non-positive; the origin is a multiple point eigenvalue in virtue of the conservation of mass, momentum, and energy. The singular part of L is given in terms of the collision frequency $\nu(\xi)$, $\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$. For a certain class of repulsive <u>hard</u> force laws (this includes all power laws with exponent higher than the Maxwellian fifth power and includes hard spheres), ν is monotone increasing in ξ and is bounded away from zero, $\nu(0) > 0$. We normalize

the time scale such that

$$\nu(0) = 1 \ , \tag{2.5}$$

thus $\nu(\xi) \geq 1$. The nonsingular part, K, of the collision operator is a bounded, compact integral operator [7].

From these properties we can predict the location of the continuous spectrum (also the conservation eigenvalues), in some cases as a consequence of proved theorems, in others as plausible conjectures. In addition to free streaming (2.1) and spatial uniformity (2.2), we consider the initial value problem for a fixed wave number k,

$$\frac{\partial f}{\partial t} + i(\xi \cdot k)f = L[f] \ , \tag{2.6}$$

steady flow

$$\xi \cdot \frac{\partial f}{\partial x} = L[f] \ , \tag{2.7}$$

and separated time

$$i\omega f + \xi \cdot \frac{\partial f}{\partial x} = L[f] \ . \tag{2.8}$$

We have already identified the continuum in free flow as the entire imaginary axis (real ω), Fig. 1. Under spatial uniformity (2.2), the continuum is real and consists of all values taken by $-\nu(\xi)$, Fig. 2. The initial value problem (2.6) has as its continuum the values taken by $-\nu(\xi) - i\xi \cdot k$, Fig. 3a for small k and

Fig. 3b for large k. The perturbed multiple eigenvalues

Fig. 1

Fig. 2

in Fig. 3b have probably disappeared to the left of the continuum.

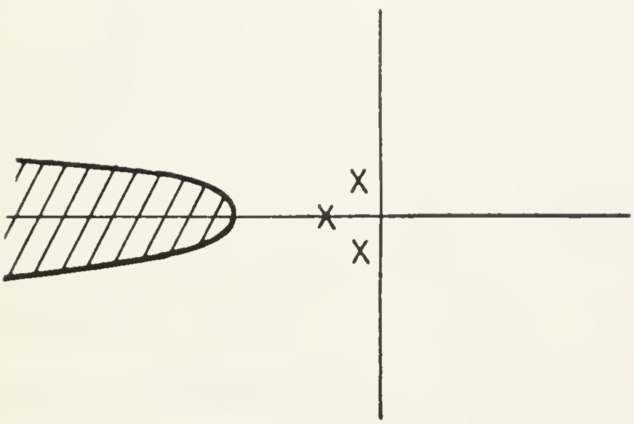

Fig. 3a

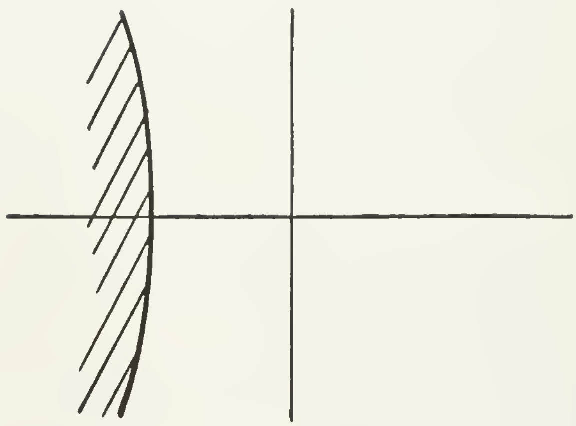

Fig. 3b

In a steady flow (2.7), the continuum covers the entire real axis, Fig. 4a, since it takes all values

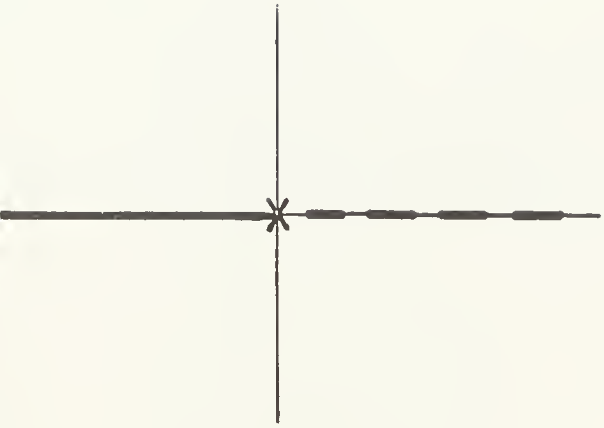

Fig. 4a

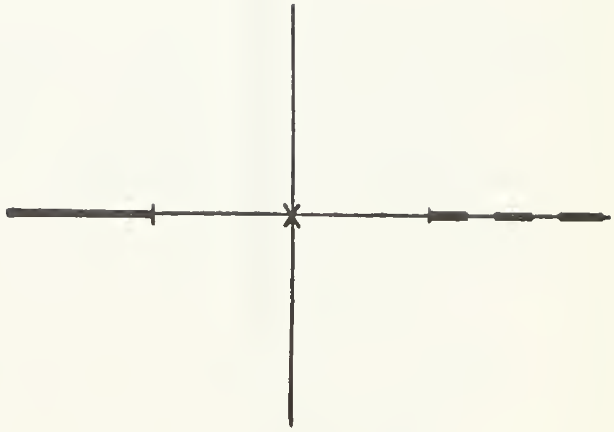

Fig. 4b

$-\nu(\xi)/\xi_1$ (ξ_1 is the component in the differentiated direction, say x_1). We have drawn half of the continuum dashed. In a half space, boundedness at infinity should eliminate half of the spectrum. The point eigenvalues at the origin are not perturbed and correspond to three exact solutions: f = const., T = linear, v = linear, corresponding to a trivial solution, simple heat flow, and simple shear flow (the first Chapman-Enskog or Navier-Stokes term is exact). For the case of hard spheres only, $\nu(\xi)/\xi_1$ is bounded away from the origin, and the spectrum is as shown in Fig. 4b.

With time separated (2.8), the steady flow spectrum

becomes complex, Fig. 5a for small ω and Fig. 5b for large ω; for hard spheres the spectrum is as in Fig. 3 (plus a mirror image spectrum in the right half plane).

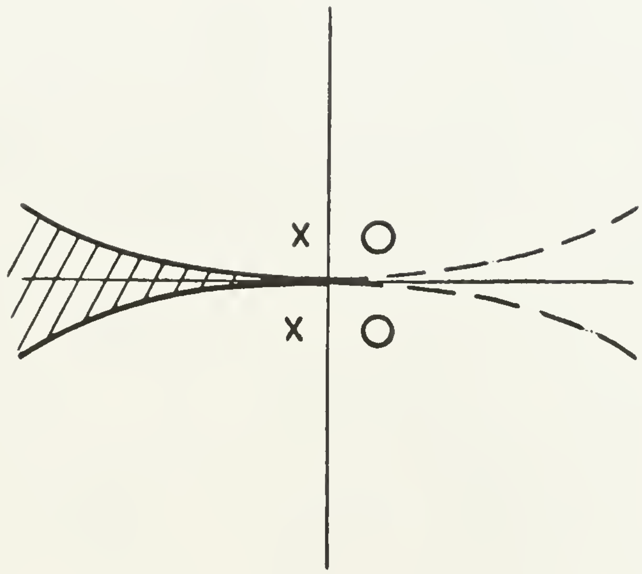

Fig. 5a

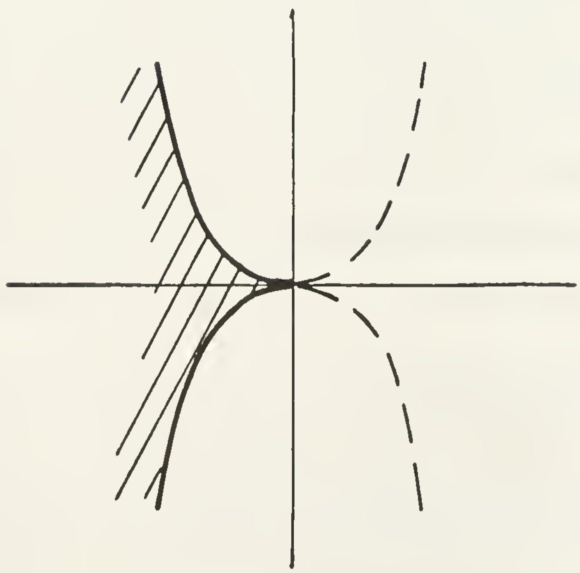

Fig. 5b

Since ν is a function of $\xi_1^2 + \xi_2^2 + \xi_3^2$ and only the component ξ_1 multiplies $\partial f/\partial x_1$, the complex spectra are not curves but entire regions. The spectrum covers the shaded areas in Figs. 3 and 5.

For a relaxation model equation (ν is constant and K is a projection), the spectrum in the spatially uniform problem (2.2) is the single infinitely degenerate point $-\nu$; in the initial value problem corresponding to (2.6), the spectrum in Fig. 3 becomes a vertical line through $-\nu$; for a steady flow (2.7), Fig.4a is unchanged; and with time separated (2.8), the spectrum in Fig. 5 consists of two straight lines through the origin (but not the enclosed region). The spectral resolution (both discrete and continuum) is explicit in many problems for the relaxation model. Several steady state problems have been solved with this technique by Cercignani [14], [15].

In the cases where the spectrum falls on the real axis, the identification of the continuum follows from known theorems, and where it is complex from plausible conjectures (the operators are not self-adjoint). The indicated perturbation of the point eigenvalues is a

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14. C. Cercignani, Ann. d. Phys. 20, 219 (1962).

15. C. Cercignani and F. Sernagiotto, "Rayleigh's Problem at Low Mach Numbers According to Kinetic Theory," to appear in Proceedings of Fourth International Symposium on Rarefied Gas Dynamics, Toronto, 1964.

plausible conjecture except in the case of hard spheres where it has been proved for sufficiently small k [16]. The mathematical proof of the asymptotic approximation of Boltzmann solutions to fluid dynamics is independent of the existence of such perturbed normal modes [9].

The behavior after a long time or far from a boundary is obtained from the least damped part of the spectrum. The part of the continuum near the origin always comes from large values of ξ_1. In Fig. 3a the dominant effect is clearly given by the point eigenvalues. In Fig. 4 the ultimate behavior (linear in x) is given by the multiple eigenvalue, but the approach to this asymptotic state is not simple. In Fig. 5a the continuum dominates very far from the wall, but this contribution is small in amplitude (since it is excited by the tail of the distribution); thus the more highly damped eigenvalues dominate for a while if ω is not too large.

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16. J. A. McLennan, Physics of Fluids 8, 1580 (1965).

3. Approach to the Boundary

There is a nonuniform approach of the distribution function to its value at the wall because of the slow particles which have a very small mean free path. We assume here that a solution exists, and we proceed to find its properties. Consider a steady flow, and for simplicity a one-dimensional problem. The steady flow equation (2.7) can be written in the integral form

$$f_+ = \overline{f}_+ \exp(-\nu x/\xi_1) + \frac{1}{\xi_1}\int_0^x \exp(-\nu(x-y)/\xi_1)K[f(y)]dy \tag{3.1}$$

for particles emerging from the wall ($\xi_1 > 0$), and a similar expression for f_- ($\xi_1 < 0$). Noting the identity

$$\frac{\nu}{\xi_1}\int_0^x \exp(-\nu(x-y)/\xi_1)dy = 1 - \exp(-\nu x/\xi_1) \ , \tag{3.2}$$

we can interpret f_+ at x as a mean value of $\overline{f}_+$ and the values of K/ν distributed along $0 < y < x$ with total weight one. The relative weights assigned to $\overline{f}_+$ and K/ν depend on x and ξ. As $x \to 0$ the weight concentrates more on $\overline{f}_+$, and f_+ approaches its boundary value $\overline{f}_+$. But the approach is not uniform in ξ_1; if $\xi_1 \to 0$ as $x \to 0$ there remains a finite deviation of f_+ from $\overline{f}_+$. This nonuniformity will not affect any quantity integrated over ξ since the deviation is finite and is concentrated on a

set of small measure as $x \to 0$. Therefore, if we assume that f_- approaches a limiting value (whether uniformly or not), the integrated expression $K[f]$ will approach a definite limiting function as $x \to 0$. And even if $\bar{f} = (\bar{f}_+, \bar{f}_-)$ is discontinuous at $\xi_1 = 0$,* $K[f]$ will be continuous. For purposes of the limit $x \to 0$, $\xi_1 \to 0$ we need only consider

$$\bar{k}(\xi_2,\xi_3) = \lim_{\substack{x \to 0 \\ \xi_1 \to 0}} K[f]/\nu \tag{3.3}$$

The deviation of f_+ from $\bar{f}_+$ for small x and ξ_1 is given by

$$f_+ - \bar{f}_+ = (\bar{k} - \bar{f}_+)[1 - \exp(-\bar{\nu}x/\xi_1)] \tag{3.4}$$

where $\bar{\nu} = \nu(0,\xi_2,\xi_3)$. The value of $f_+ - \bar{f}_+$ at any (ξ_2,ξ_3) depends on the ratio x/ξ_1. It is an elementary exercise to verify that

$$\int_{-a}^{+a} [1 - \exp(-\bar{\nu}/x\xi_1)]d\xi_1 = 2\bar{\nu}\, x \log x + O(\bar{\nu}x) \tag{3.5}$$

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*In a plane geometry f is generally discontinuous at $\xi_1 = 0$, and in an exterior domain across a line of sight cone. But in a convex interior domain f is continuous even in free flow (e.g., with a varying wall temperature).

Thus any macroscopic moment of $f_+ - \bar{f}_+$ is $O(x \log x)$, whereas the contribution to the integral from finite ξ_1 is only $O(x)$.

There are several conclusions. The convergence of f_+ to its boundary value is nonuniform in a small velocity region ξ_1 0 of order x, Fig. 6. There is no

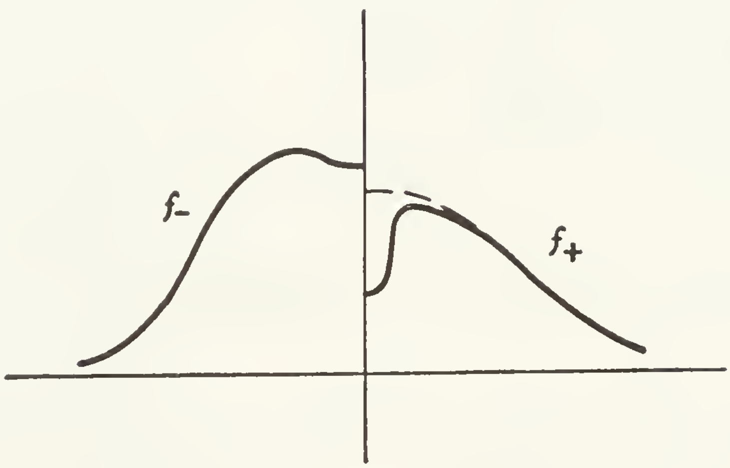

Fig. 6

reason to expect the approach of f_- to be nonuniform. The approach of a macroscopic moment to its boundary value is uniform, $x \log x \rightarrow 0$, but the derivative is unbounded. This infinity is in addition to a possible discontinuity such as temperature or velocity slip, Fig. 7. If two plates at different temperature or velocity are separated by a distance L which is small compared to the mean free path, the incoming distribution f_- will have a

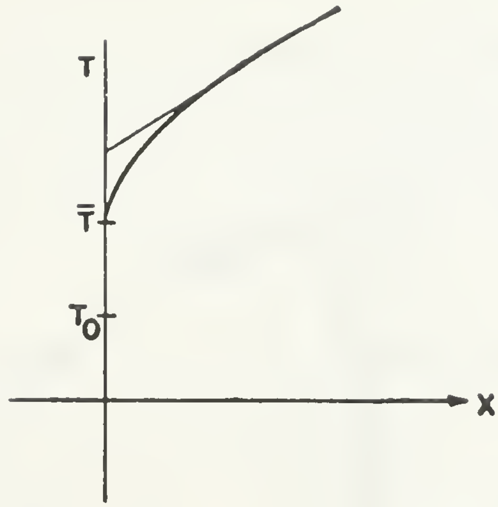

Fig. 7

disturbed region of finite width $O(L)$, and f_+ will have a boundary layer of width $O(x)$ as $x \to 0$. A macroscopic moment will have infinite slope at each wall and a slope $O(\log L)$ at the midplane.

With a time factor $i\omega$, the analysis of f_+ is essentially the same except that ν is replaced by $\nu + i\omega$. The oscillatory factor $\exp(-i\omega x/\xi_1)$ yields an additional nonuniform limit with very rapid oscillations concentrated at $\xi_1 = 0$. A typical f_+ would look somewhat as shown in Fig. 8. If $\omega >> 1$, the oscillatory boundary layer with a width $\delta\xi_1 \sim \omega x$ dominates. If $\omega << 1$, the previous boundary layer of thickness $\delta\xi_1 \sim x$ dominates, but there is an oscillatory sublayer of still smaller thickness ωx. This complex nonuniform limiting behavior is not necessarily physically important, but it must be taken under careful consideration in evaluating mathematical approximations.

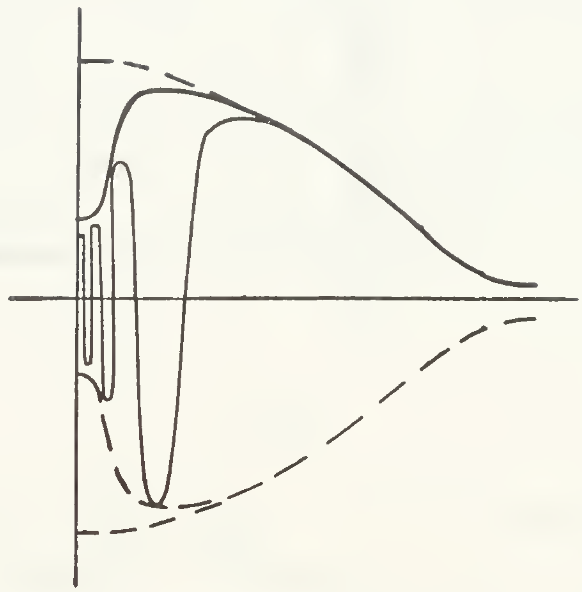

Fig. 8

4. Existence in the Small

Consider a one-dimensional problem between two parallel plates for the separated time equation (2.8). We suppose that the emerging distribution $f_+ = \overline{f}_+$ is given at the left and $f_- = \overline{f}_-$ is given at the right (for conversion of these results to a more conventional boundary condition which relates f_+ to f_- at a wall, see Sec. 6). This problem is essentially the same one that has been

treated by Willis [17]; the presence of the additional term $i\omega f$ offers no increase in difficulty, but we shall present a different proof. First we rewrite the Boltzmann equation as a pair of integral equations

$$
\begin{aligned}
f_+ &= \bar{f}_+ \exp\left[-\frac{(\nu+i\omega)x}{\xi_1}\right] + \frac{1}{\xi_1}\int_0^x \exp\left[-\frac{(\nu+i\omega)(x-y)}{\xi_1}\right] K\,dy \\
f_- &= \bar{f}_- \exp\left[-\frac{(\nu+i\omega)(x-L)}{\xi_1}\right] + \frac{1}{\xi_1}\int_L^x \exp\left[-\frac{(\nu+i\omega)(x-y)}{\xi_1}\right] K\,dy
\end{aligned}
\tag{4.1}
$$

We have $\xi_1 > 0$ in the first equation and $\xi_1 < 0$ in the second. For a steady flow we merely set $\omega = 0$. We shall wish to iterate these equations for small L. To make estimates, we take absolute values

$$
\begin{aligned}
|f_+| &< |\bar{f}_+| \exp\left[-\frac{\nu x}{\xi_1}\right] + \frac{1}{\xi_1}\int_0^x \exp\left[-\frac{\nu(x-y)}{\xi_1}\right] |K|\,dy \\
|f_-| &< |\bar{f}_-| \exp\left[-\frac{\nu(L-x)}{|\xi_1|}\right] + \frac{1}{|\xi_1|}\int_x^L \exp\left[-\frac{\nu(y-x)}{|\xi_1|}\right] |K|\,dy
\end{aligned}
\tag{4.2}
$$

and we see that ω disappears.

We cannot drop the exponentials to simplify the

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17. D. Roger Willis, "The Effect of the Molecular Model on Solutions to Linearized Couette Flow with Large Knudsen Number," in Rarefied Gas Dynamics, ed. L. Talbot, Academic Press (1961).

estimates, for this would leave the non-integrable factor $1/\xi_1$. We know that K is bounded in an absolute norm,

$$|K[f]| < \kappa_0 \max|f| \; . \tag{4.3}$$

But we cannot use this simple estimate to prove convergence together with the interpretation of (4.1) as a mean value of f and K/ν because κ_0 is larger than one. We cannot use this estimate even for arbitrarily small L because the weight $[1 - \exp(-\nu x/\xi_1)]$ which multiplies K/ν is arbitrarily close to one if ξ_1 is small. But if L is small, this weight is small for <u>most</u> values of ξ_1. We therefore combine a maximum and L_2 norm to show that iteration of (4.2) about the second term on the right converges for small L.

We use two properties of the operator K[f] from [7]

$$(1+\xi^2)^{3/2}|K[f]| < \kappa_1 \max_{\xi}(1+\xi^2)|f| \tag{4.4}$$

$$|K[f]| < \kappa_2 \left[\int f^2 d\xi\right]^{1/2} \tag{4.5}$$

Both κ_1 and κ_2 are larger than one. From the first relation we know that application of K reduces f for large ξ, specifically for $(1+\xi^2)^{1/2} > \kappa_1$; but K can increase f at smaller values of ξ. It is an elementary consequence of (4.4) and (4.5) that if

$$|f| < (1+\xi^2)^{-1}$$

$$\int f^2 d\xi < (8\kappa_1^2\kappa_2)^{-2} \equiv a^2 \qquad (4.6)$$

are both satisfied, then

$$|K[f]| < \tfrac{1}{2}(1+\xi^2)^{-1} . \qquad (4.7)$$

We can, in addition, find a value L such that for $|Kf|$ satisfying (4.7),

$$\int [1 - \exp(-\nu L/\xi_1)]^2 \, |K[f]|^2 d\xi < \tfrac{1}{4} a^2 . \qquad (4.8)^*$$

If $\max_x |f|$ satisfies (4.6), we say it has norm [1,a]. From the estimates (4.7) and (4.8), the integral terms in (4.2) have norm [1/2, 1/2a] if f has norm [1,a]. The iteration is therefore contracting and converges uniformly in both the maximum and L_2 norms.

The significance of the values of the various constants depends on the fact that the Boltzmann equation has been made dimensionless, distance with respect to mean free path, time with respect to mean collision time [approximately, by $\nu(0) = 1$], and velocity with respect to mean thermal velocity. In these variables the constants κ_o, κ_1, κ_2 as well as the critical value of L are all of order one. The behavior of

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*The proof of this inequality is not difficult; a similar result will be proved as the Lemma of Sec. 5.

the solution in ω is, of course, completely lost by this method of estimation.

5. Existence in a Half Space

We consider the equation

$$i\omega f + \xi_1 \frac{\partial f}{\partial x} + \nu f = K[f] \tag{5.1}$$

in the half space $x \geq 0$. To obtain existence for unlimited x we cannot afford to take absolute values and ignore the oscillations induced by ω (in particular, there is no existence theorem for steady solutions, $\omega = 0$, in the half space $x > 0$). We can use ω effectively only by taking a Fourier or Laplace transform in x. The parameter ω will reduce the amplitude of the transformed f in frequency space except at certain resonant points. We cannot afford to use a maximum norm alone, for K has norm larger than one and ω does not reduce the amplitude uniformly. An L_2 norm will be reduced by ω and, as in the previous section, can be used to reduce the maximum norm as well.

Assuming that f is bounded as $x \longrightarrow \infty$, we can rewrite (5.1) as

$$f_+ = \bar{f}_+ \exp(-\alpha x) + \frac{1}{\xi_1} \int_0^x \exp[-\alpha(x-y)] K[f(\xi, y)] dy \qquad (5.2)$$

$$f_- = -\frac{1}{\xi_1} \int_x^\infty \exp[\alpha(y-x)] K[f(\xi, y)] dy \qquad (5.3)$$

where

$$\alpha = \frac{1}{\xi_1}(\nu + i\omega) \; ; \qquad (5.4)$$

ξ_1 and Re α are positive in (5.2) and negative in (5.3). The incident distribution at $x = 0$ is

$$\bar{f}_- = -\frac{1}{\xi_1} \int_0^\infty \exp[\alpha(y-x)] K dy \; . \qquad (5.5)$$

In the present formulation we shall assume that $\bar{f}_+$ is given; $\bar{f}_-$ will be determined by the condition that f be bounded. In the next section we extend this formulation to include more conventional boundary conditions relating $\bar{f}_+$ to $\bar{f}_-$.

We introduce the Laplace transform

$$\tilde{f}(\xi, s) = \int_0^\infty e^{-sx} f(\xi, x) dx \qquad (5.6)$$

and note that the transform of K[f] is simply $K[\tilde{f}]$. Dropping the tilde, we have the transformed equation

$$f = \frac{\xi_1 \bar{f} + K[f]}{\nu + i\omega + \xi_1 s} \; . \qquad (5.7)$$

We would obtain the same result by a Fourier transform of a function $f^*(x)$ where $f^* = f$ for $x > 0$ and $f^* = 0$ for $x < 0$. We shall use s for the complex variable and write $s = ik$ with real k when s is to be restricted to the imaginary axis. To invert the transform we require that f be analytic for $\mathrm{Re}(s) > 0$ and $f \longrightarrow 0$ for large s.

The denominator in (5.7) vanishes if $-s = (\nu+i\omega)/\xi_1 = \alpha$. The pole is in the left half plane if $\xi_1 > 0$ and in the right half plane if $\xi_1 < 0$. If $\xi_2 = \xi_3 = 0$, $\nu = \nu(\xi_1)$, this pole lies exactly on the curve (Fig. 5) which was described as the boundary of the continuous spectrum; if $\xi_2^2 + \xi_3^2 > 0$, the pole is to the left of the spectral boundary in the left half plane and to the right of this curve on the right. To solve the problem as posed, the singularities in the right half plane must be removed. To this end we set

$$\overline{f}_- = -\frac{1}{\xi_1} K[f(\xi, -\alpha)] \tag{5.8}$$

for $\xi_1 < 0$ and split (5.7) into

$$f_+ = \frac{\xi_1 \overline{f}_+ + K[f]}{\nu+i\omega + s\xi_1} \tag{5.9}$$

$$f_- = \frac{K[f(\xi,s)] - K[f(\xi,-\alpha)]}{\xi_1(s+\alpha)} \tag{5.10}$$

It is in this formulation, (5.9) and (5.10), that we shall

solve the problem; $\overline{f}_+$ is assumed to be known, and $\overline{f}_-$ can be computed from (5.8) once f is known.

We introduce the function class C as follows. The domain D_+ consists of all real ξ_2, ξ_3, $\xi_1 \geq 0$ together with Re $s \geq 0$. The function $f_+(\xi, s)$ is in C_+ if $(1+\xi^2)f_+$ is continuous and bounded in D_+; and if f_+ is analytic in Re $s > 0$ for each ξ (it is continuous to $s = ik$); and if $|f| < C/|s|$ as $s \rightarrow \infty$ for each ξ (C need not be uniform). The class C_- is similarly defined for f_- and D_- with $\xi_1 \leq 0$. We say that f is in C if $f_\pm$ are in $C_\pm$. There is no implication that f is continuous; the $f_\pm$ may have distinct limits as $\xi_1 \rightarrow \pm 0$.

To start an iteration of (5.9) and (5.10) we set $f = f^{(o)}$

$$
\begin{aligned}
f_+^{(o)} &= \xi_1 \overline{f}_+ / (\nu + i\omega + s\xi_1) \\
f_-^{(o)} &= 0
\end{aligned}
\tag{5.11}
$$

We remark that for $s = ik$

$$
|\nu + i(\omega + k\xi_1)| > [1 + (\omega + k\xi_1)^2]^{1/2} > 1 \tag{5.12}
$$

Taking $\overline{f}_+$ such that $(1+\xi^2)\xi_1\overline{f}_+ < \infty$, we verify that $f_+^{(o)}$ is in C_+. It is continuous in (ξ, s), it is bounded on the imaginary axis, and it is analytic and decays as $1/|s|$ in the right half plane.

Now setting

$$f_+^{(n+1)} = f_+^{(o)} + \frac{K^{(n)}(\xi,s)}{\nu + i\omega + s\xi_1} \tag{5.13}$$

$$f_-^{(n+1)} = \frac{K^{(n)}(\xi,s) - K^{(n)}(\xi,-\alpha)}{\nu + i\omega + s\xi_1} \tag{5.14}$$

we verify that all iterates remain in C. For induction we assume this property for $f^{(n)}$. From (4.4) we see that $(1+\xi^2)K^{(n)}$ is bounded. It is continuous and analytic in s because the kernel of K is continuous and bounded (except for an integrable singularity at the origin [7]). Using (5.12) we know that $(1+\xi^2)f_+^{(n+1)}$ is bounded on s = ik and $f_+^{(n+1)} \to 0$ as $|k| \to \infty$. Therefore $f_+^{(n+1)}$ is in C_+. We may use the same estimates for $K^{(n)}(\xi,s)$ and $K^{(n)}(\xi,-\alpha)$ in (5.14). In particular, $f_-^{(n+1)}$ is analytic for each ξ; the pole is canceled if $\xi_1 \neq 0$, and there is no pole if $\xi_1 = 0$. Also $f_-^{(n+1)}$ is bounded on s = ik and it approaches zero as $|k| \to \infty$; for $\xi_1 \neq 0$ this is explicit, and for $\xi_1 = 0$, $K^{(n)}(\xi,-\alpha) = 0$ and $K^{(n)}(\xi,ik) \to 0$ with k. It remains only to establish continuity in (s,ξ) for $s \sim \infty$ and $\xi_1 \sim 0$. This follows from uniform convergence on s = ik as $\xi_1 \to 0$. In particular, we observe no difficulty at the origin, s = 0, even though this belongs to the spectrum, because this part of the spectrum arises for large ξ where all functions are uniformly small.

To show convergence of the iterates we introduce the same norm as in the last section, viz., that f has norm

$[1,a]$ if

$$\begin{aligned} &|f| < (1+\xi^2)^{-1} \\ &\int f^2 d\xi < a^2 \end{aligned} \tag{5.15}$$

and if each of $f_{\pm}$ has norm $[1, a/\sqrt{2}]$. We set

$$a = c^3/\kappa_1^2\kappa_2 \tag{5.16}$$

where $c < 1$ is a positive constant to be fixed later [κ_1 and κ_2 are given in (4.4) and (4.5)]. We now prove the

Lemma: Assume that f has norm $[1,a]$ and set

$$g = [\nu + i(\omega + k\xi_1)]^{-1}K[f] \; . \tag{5.17}$$

Then, for $\omega > \omega_o$ (depending on κ_1, κ_2, c), g has norm less than $[c,ca]$.

First we show that $(1+\xi^2)|g| < c$. Using the two bounds (4.4) and (4.5) in turn, we have

$$(1+\xi^2)|g| < (1+\xi^2)^{-1/2}\,\kappa_1$$

$$(1+\xi^2)|g| < (1+\xi^2)\kappa_2\, a$$

Setting $1+\xi_o^2 = \kappa_1^2/c^2$, the first estimate is less than c for $\xi > \xi_o$ while the second is less than c for $\xi < \xi_o$. To complete the lemma we must show that $\int g^2 d\xi < c^2a^2$. We have, using (4.4) and (5.12)

$$\int g^2 d\xi < \kappa_1^2 \int (1+\xi^2)^{-3}[1 + (\omega + k\xi_1)^2]^{-1} d\xi$$

$$= \pi\kappa_1^2 \int_{-\infty}^{\infty} [1 + (\omega + k\xi_1)^2]^{-1} d\xi_1 \int_0^{\infty} [1 + u + \xi_1^2]^{-3} du$$

$$= \frac{1}{2}\, \pi\kappa_1^2 \int_{-\infty}^{\infty} (1 + \xi_1^2)^{-2} [1 + (\omega + k\xi_1)^2]^{-1} d\xi_1$$

This can be integrated explicitly, but it is easier to estimate. For the limited range of integration where $|\omega + k\xi_1| > \mu$, writing $[1 + (\omega + k\xi_1)^2] > \mu^2$ we obtain the upper bound

$$\frac{1}{4}\, \pi^2 \kappa_1^2/\mu^2 \ .$$

For the remainder, $|\omega + k\xi_1| < \mu$, we write $[1 + (\omega + k\xi_1)^2] > 1$ and $(1 + \xi_1^2) > [1 + (\omega-\mu)^2/k^2]$ to obtain (after maximizing with respect to k)

$$\frac{8\pi}{25}\, \kappa_1^2 \, \frac{\mu}{|\omega - \mu|} \ .$$

Together,

$$\int g^2 d\xi < \kappa_1^2 \left[\frac{\pi^2}{4} \frac{1}{\mu^2} + \frac{8\pi}{25} \frac{\mu}{|\omega - \mu|} \right]$$

Setting

$$\begin{aligned} \omega_o &= \mu + 32\mu^3/25\pi \\ \mu^2 &= \pi^2 \kappa_1^2/2a^2c^2 \ , \end{aligned} \tag{5.18}$$

we have $\int g^2 d\xi < a^2c^2$ for $\omega > \omega_o$, proving the lemma.

We are now ready to show convergence of the iteration for $\omega > \omega_o$. If $f_+^{(n)} - f_-^{(n-1)}$ has norm $[1, a/\sqrt{2}]$ then $f_+^{(n+1)} - f_+^{(n)}$ has norm $[c, ca/\sqrt{2}]$. Similarly, the norm $[1, a/\sqrt{2}]$ for $f_-^{(n)} - f_-^{(n-1)}$ induces a norm $[2c, \sqrt{2}\, ca]$ for $f_-^{(n+1)} - f_-^{(n)}$. The iteration is contracting if we take $c < 1/2$ and take $f_+^{(o)}$ of finite norm. It is, of course, sufficient to make these estimates on the imaginary axis $s = ik$ since all the iterates are analytic.

By the uniform convergence we have proved the existence of a solution $f(\xi,s)$ which is in C. The inverse transform exists and yields a bounded solution in x. The iterations also converge uniformly to the incident $\overline{f}_-$. And since the convergence factor increases with ω, the function $\overline{f}_-$ goes to zero for large ω. In other words, no particles are scattered back when ω is large. Also the first term $f_+^{(o)}$ is dominant in some sense (which we shall examine in Sec. 7) for large ω. The iteration of the Laplace transform $f(\xi,s)$ corresponds exactly to iteration of the integral equations (5.2) and (5.3) for $f(\xi,x)$.

A better bound on $\overline{f}_+$ such as $\overline{f}_+ \sim (1+\xi^2)^{-n}$ will yield corresponding estimates for the solution. The decay as $x \longrightarrow \infty$ can be estimated by integrating the inverse Laplace transform by parts. If $\overline{f}_+$ decays algebraically for large ξ, then f will decay algebraically in x [$f(\xi,s)$ has only a finite number of derivatives at $s = 0$]. If $\overline{f}_+$ has all moments (e.g., Maxwellian), then f will decay transcendentally

in x.

An existence theorem for the nonlinear Boltzmann equation can be carried out in almost the same way, using the collision term estimates of Ref. [8]. The result is surprisingly uninteresting. In ordinary gas dynamics a linearization becomes increasingly worse at high frequencies (the common linearization takes derivatives to be uniformly small). But the correct kinetic approach shows that at very high frequencies, where the fluid analysis is no longer valid, the nonlinearity becomes progressively less important. The reason is that in the Boltzmann equation the nonlinearity resides only in the collision term, whereas it is the strictly linear streaming term which dominates at high frequency. In other words, the solution decays to linearity in much less than a mean free path before collisions play a significant role. This is an important point with regard to experiment, because high frequency experiments are inevitably operated at a high excitation level in order to be able to detect the strongly damped waves.

6. Boundary Conditions

The problem that we have solved is artificial in that the emitted distribution $\overline{f}_+$ was assumed to be given outright. The more conventional formulation is to give $\overline{f}_+$ in terms of $\overline{f}_-$. The general form of such a boundary condition is

$$\overline{f}_+ = f_o + B[\overline{f}_-] \tag{6.1}$$

where f_o is given and B is a bounded linear operator; we shall give B explicitly for specular and diffuse reflection below. But the solution obtained in the last section gives $\overline{f}_-$ as a bounded linear operator on $\overline{f}_+$,

$$\overline{f}_- = R[\overline{f}_+] \, . \tag{6.2}$$

This operator is obtained explicitly in the iteration as a geometrically convergent sequence, and the norm of R goes to zero for large ω. The linear functional equation

$$\overline{f}_+ = f_o + BR[\overline{f}_+] \tag{6.3}$$

has a convergent Neumann series for sufficiently large ω, viz., for which the norm of BR is smaller than one. Therefore $\overline{f}_+$ can be found and the existence theorem applied to the general boundary condition (6.1).

In a specific problem it would be more expedient to iterate (6.1) and the Boltzmann equation together, taking

$\overline{f}_+ = f_o$ as the initial boundary condition and improving $\overline{f}_+$ and f in turn. The dominant term for very large ω would be the combined first iterate,

$$
\begin{aligned}
f_+ &= f_o \exp[-(\nu + i\omega)x/\xi_1] \\
f_- &= 0
\end{aligned}
\tag{6.4}
$$

Both the collision operator K and the boundary operator B become unessential in the limit of large ω.

With a diffuse reflection boundary condition the full nonlinear boundary distribution is

$$
\overline{F}_+ = \frac{\rho}{(2\pi RT)^{3/2}} \exp\left[- \frac{(\overline{\xi}-u)^2}{2RT}\right] \tag{6.5}
$$

where u(t) represents the oscillating wall (both normal and tangential), T(t) can also oscillate, and $\rho(t)$ must be left open to satisfy conservation of mass. We linearize according to the scheme

$$
\begin{aligned}
\rho &= \rho_o(1 + \rho_1 e^{i\omega t}) \\
T &= T_o(1 + T_1 e^{i\omega t}) \\
u &= (RT_o)^{1/2}\, u_1 e^{i\omega t}
\end{aligned}
\tag{6.6}
$$

and introduce the dimensionless velocity $\xi = \overline{\xi}(RT_o)^{-1/2}$ and the normalization

$$F = F_o(1 + \omega^{1/2} f)$$

$$\omega = \frac{1}{(2\pi)^{3/2}} \exp(-\tfrac{1}{2}\xi^2) \tag{6.7}$$

which are implicit in the derivation of the dimensionless linear Boltzmann equation which we have been using (for details see [10]). We have

$$\bar{f}_+ = \omega^{1/2}[\rho_1 + T_1(\tfrac{1}{2}\xi^2 - \tfrac{3}{2}) + u_1\xi_1 + u_2\xi_2] , \tag{6.8}$$

where u_1 is the normal component and u_2 the tangential component (taken for simplicity in the direction ξ_2). The conservation of mass,

$$\int_{\xi_1>0} \xi_1 \bar{f}_+ \omega \, d\xi + \int_{\xi_1<0} \xi_1 \bar{f}_- \omega \, d\xi = 0 \tag{6.9}$$

serves to eliminate ρ_1 and gives the boundary condition

$$\begin{aligned} \bar{f}_+ = {} & \omega^{1/2}\left\{u_1[\xi_1 + (\pi/2)^{1/2}] + u_2\xi_2 + T_1[\tfrac{1}{2}\xi^2 - 2]\right\} \\ & - (2\pi\omega)^{1/2} \int_{\xi_1<0} \xi_1 \bar{f}_- \omega \, d\xi . \end{aligned} \tag{6.10}$$

The first term [f_o in (6.1)] is linear in the given oscillation amplitudes u_1, u_2, and T_1 and is a polynomial in ξ times $\omega^{1/2}$. The operator B in (6.1) is in this case a simple projection (first moment of $\bar{f}_-$). For a simple normal

oscillation of the wall we set $u_2 = T_1 = 0$, for a simple shear $u_1 = T_1 = 0$, and for a stationary but heated wall $u_1 = u_2 = 0$. Since the formulation is linear, each elementary problem can be solved separately.

A similar linearization of the specular reflection boundary condition of a normally oscillating wall (neither shear nor temperature variation can couple to the gas with specular reflection) yields the boundary condition

$$\overline{f}_+(\xi_1) = \omega^{1/2}\xi_1 u_1 + \overline{f}_-(-\xi_1) \tag{6.11}$$

The operator B in this case is the identity.

7. High Frequency Behavior

The dominant feature at high ω is that f is rapidly oscillating in ξ as well as in x, Fig. 9. The first

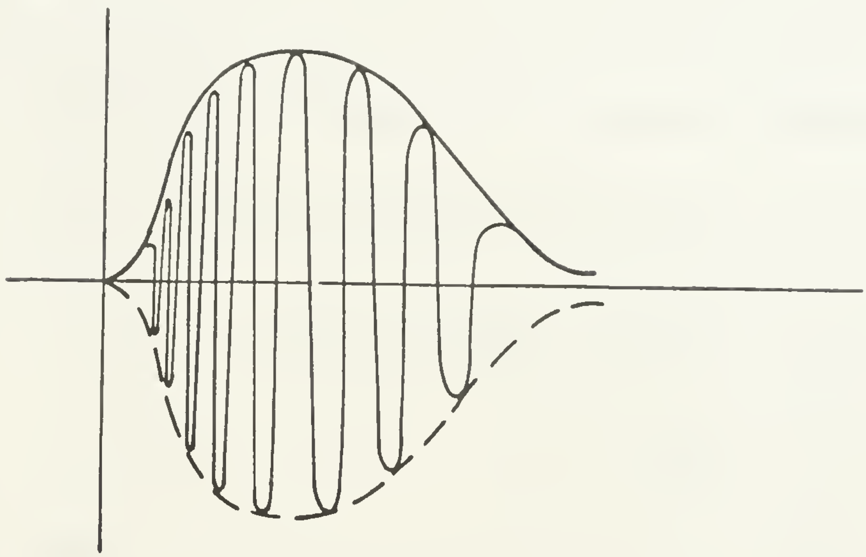

Fig. 9

iterate is given by

$$
\begin{aligned}
f_+ &= f_o \exp[-(\nu + i\omega)x/\xi_1] \\
f_- &= 0
\end{aligned}
\tag{7.1}
$$

where, as in (6.4), we have taken the leading term in the boundary condition as well as in the Boltzmann equation. If we increase ω at fixed x and ξ, f_+ oscillates but with fixed amplitude $f_o \exp(-\nu x/\xi_1)$. We shall verify that at any fixed x and ξ, $\xi_1 \neq 0$, the error in taking the leading

term (7.1) is small compared to this amplitude factor $f_o \exp(-\nu x/\xi_1)$. Any moment or other integrated quantity over f_+ is small; algebraically as ω^{-n} if f_o decays algebraically for large ξ, and transcendentally small in ω if f_o decays transcendentally in ξ (e.g., Maxwellian). Thus $K[f_o]$ and the consequent error in the formula (7.1) are small compared to the term which is kept,

$$\lim_{\omega \to 0} |f_+ - f_o \exp[-(\nu + i\omega)x/\xi_1]| = 0$$

$$\lim_{\omega \to 0} |f_+| = |f_o| \exp(-\nu x/\xi_1) \qquad (7.2)$$

$$\lim_{\omega \to 0} |f_-| = 0$$

This result is interesting in that x can be several mean free paths. Collisions do have an effect in cutting down the boundary emission f_o by the factor $\exp(-\nu x/\xi_1)$, but there is no compensating term from the creation operator K[f] to this order. The reason is simply that νf is singular and K[f] is regular; any averaging over f reduces the amplitude because of the rapid oscillations.

The answer is much more subtle if, instead of the error in f, we look at the error in a moment of f. In particular, it is a moment which is usually the experimental observable. For a Maxwellian f_o, any moment will be transcendentally small in ω. It is plausible that the next

iteration, which first involves evaluation of K[f] (comparable to evaluating a moment) and then another average of K with a highly oscillating integrand [cf. (4.1)], will give a contribution smaller than the leading term. For an algebraic decay of f_o this can be demonstrated. But for the much more relevant exponentially decaying f_o, this estimate presents great difficulty. The reason is that the leading term (7.1) cannot approximate the actual solution uniformly for small $\xi_1 \sim 1/\omega x$. Even for arbitrarily small x we have seen that f_+ does not uniformly approximate $\overline{f}_+$ (Sec. 3). If the moment is transcendentally small compared to the fluctuating amplitude of f, we cannot eliminate the small region $\xi_1 \sim 1/\omega x$ from the integration without estimating it accurately. Although this is a gap in the adoption of the leading term (7.1) for integration purposes, the area of doubt in the estimate, viz., small ξ_1, is where $f_+(\xi)$ oscillates most rapidly, Fig. 9; it is quite plausible that this difficulty is technical rather than actual.

Summarizing, the amplitude of $f^{(o)}$ remains finite as $\omega \longrightarrow 0$ and is clearly dominant asymptotically; the estimate is more subtle for a moment of f but is very plausible.

We shall now accept the accuracy of (7.1) and evaluate its moments (e.g., density and pressure) for possible experimental comparison. For either diffuse or specular reflection, the function f_o has the form

$$f_o \sim \xi_1 e^{-\frac{1}{4}\xi^2} \quad \text{or} \quad \xi^2 e^{-\frac{1}{4}\xi^2} \tag{7.3}$$

for large ξ, depending on whether the wall is moving or is given a temperature oscillation. Before integrating, f is multiplied by 1 or ξ^2 depending on whether density or pressure is measured. Thus the function to be evaluated has the form

$$Q = \int \xi^n \exp[-\tfrac{1}{4}\xi^2 - \nu x/\xi_1 - i\omega x/\xi_1] d\xi \tag{7.4}$$

where n takes values between 1 and 4. The collision frequency for large ξ has the behavior

$$\nu(\xi) \sim \nu_1 \xi^\alpha \tag{7.5}$$

where ν_1 is of order one and $0 < \alpha < 1$ ($\alpha = 0$ for Maxwellian fifth power and $\alpha = 1$ for hard spheres).

Any point of steepest descent is found at $\xi_2 = \xi_3 = 0$; thus we can consider the integration (7.3) in one dimension and ignore the distinction between ξ and ξ_1. A point of steepest descent is a stationary point of

$$n \log \xi - \frac{1}{4}\xi^2 - \frac{\nu_1 x}{\xi^{1-\alpha}} - \frac{i\omega x}{\xi} . \tag{7.6}$$

The dominant contribution for large ω is free flow, given by the second and fourth terms (see also [6]),

$$\xi_o^3 = 2i\omega x . \tag{7.7}$$

The correct critical points are

$$\xi_o = (2\omega x)^{1/3}\left(\frac{1}{2}\, i \pm \frac{1}{2}\sqrt{3}\right) \tag{7.8}$$

and the path of integration for ξ is as shown in Fig. 10.

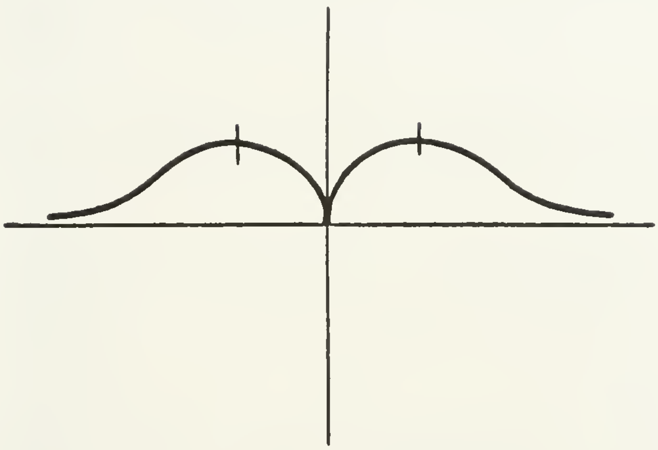

Fig. 10

The dominant asymptotic expression for Q is oscillatory in ωx but with an amplitude given by

$$\log Q \sim -\frac{3}{8}\,(2\omega x)^{2/3} \; . \tag{7.9}$$

The wavelength of the oscillation of Q is locally

$$\lambda \sim \frac{4\pi}{\sqrt{3}}\left(\frac{2x}{\omega^2}\right)^{1/3} \tag{7.10}$$

which is short, but slowly lengthens with x.

If x is considered to be fixed as ω increases, the

next order term taken from (7.5) is $n \log \lambda$ and the effect of collisions only comes in at third order. To see the effect of collisions more simply, we let x grow as $\omega^{1/2}$ which brings in $\nu(\xi)$ at second order. We obtain

$$\log Q \sim -\frac{3}{8}(2\omega x)^{2/3} - \nu_1 x(2\omega x)^{-(1-\alpha)/3}\cos[5\pi(1-\alpha)/6] \tag{7.11}$$

(valid for any scaling of x which makes $\nu x/\xi$ dominate $\log \xi$) and

$$\log Q \sim -\frac{3}{8}(2)^{2/3}\omega - \nu_1 2^{-(1-\alpha)/3}\cos[5\pi(1-\alpha)/6]\omega^{\alpha/2} \tag{7.12}$$

for $x = \omega^{1/2}$. An experimental confirmation would have to detect two terms in a plot of $\log Q$. The first, linear in ω, would be a confirmation of the Maxwellian distribution. The second, proportional to $\omega^{\alpha/2}$ would give a measurement of the force law exponent α and coefficient ν_1 for large velocity ξ.

Taking a second term in the expansion with x fixed, we have

$$\log Q \sim -\frac{3}{8}(2\omega x)^{2/3} + \frac{1}{3} n \log \omega . \tag{7.13}$$

Observation of the second term with this scaling would show the deviation from a Maxwellian evidenced by the number n.

In summary, any direct measurement of the distribution function would give a nontrivial lowest order result involving the entire function $\nu(\xi)$ and the boundary condition through the expected variation $f_o \exp(-\nu x/\xi_1)$. Measurement of a moment of f gives information about the boundary condition if we keep x = constant and information about $\nu(\xi)$ for large ξ if we scale $x \sim \omega^{1/2}$. All measurements will be amplitudes of rapidly oscillating fluctuations in x, ξ, ω as well as t.

We do not attempt to compare these results with experiment. The lowest order free flow expression (7.9) has apparently been observed experimentally [18, 19, 6]. Since the dimensional x and ω vary as p (pressure) and 1/p respectively, in a given experiment p must be varied and in addition either x or ω must be varied in order to be able to observe the second term in (7.12).

We remark that an approximation such as that made by Kahn and Mintzner [6], but with our $f^{(o)}$ as the starting point rather than free flow, might yield very accurate high frequency data.

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